A Correction to Whipple's law for Ion-Trap Current

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Abstract

We have analyzed a phenomenon heretofore ignored in the analyses of ion traps, which are used to determine ion temperature, among other plasma parameters, in planetary ionospheres: ions that are rejected by the trap perturb the plasma well ahead of the Debye sheath at the front of the trap.The determination of the perturbed plasma flow is found to depend on the fact that the ionospheric plasma be stable to quasineutral, ion-acoustic perturbations.
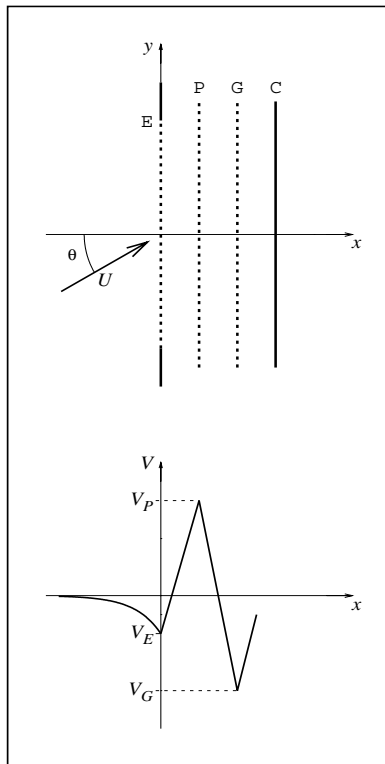
Correction to Whipple's law

Fig. 1

In hypersonic plasma flows past ion-rejecting bodies, the disturbances are not confined to thin Debye sheaths. Ions missing from the wake behind the body are those perturbing the plasma far ahead. This phenomenon was noticed in both experiments and numerical calculations [[1]], and may affect positively biased satellites (such as the one used by the TSS-1R tether to collect electrons), as well as ion traps [[2]].

Ion traps are ionospheric probes [[3]] used on board satellites, rockets, the Shuttle and in other planetary ionospheres [[4]]. A trap is a multigrid electrostatic probe (Fig.1):

- The entrance grid E in the spacecraft wall is at potential $V_E < 0$ relative to the undisturbed plasma. It repels incoming electrons.

- The retarding grid P, biased at potential $V_P > 0$, rejects some incoming ions.

- The collector C at the back collects ions that get past P.

- The suppressor grid(s) G is highly negative and turns back energetic electrons. It inhibits photo and secondary emission from C.

The ion current I reaching C is registered as a function of V_P. This allows determining plasma density, ion drifts and composition and ion temperature T_i (a quantity important for establishing the energy budget of the upper atmosphere) [[5]].

The ordering of characteristic lengths in the Earth's ionosphere simplifies the analysis of traps. We have mean free path $>>$ ion thermal gyroradius $\sim$ satellite size $>>$ trap width ($\sim$ 10 cm). Debye length λ_D and distance between grids $E - P$ ($\sim$ 1 cm) are smaller than trap width. Also, the ion thermal velocity is smaller than the spacecraft speed U. This allowed Whipple to use an onedimensional approach to derive a relation $I(V_P)$ for fitting trap data [[6]].

Discrepancies between results from traps and other instruments led to reexamining Whipple's (ideal) trap law. It was found that several effects (nonplanar sheaths, finite trap width, space charge inside the trap, energy-dependent grid transparencies, nonuniformity of potential in grid planes) may invalidate Whipple's law [[7]]. This led to a better trap design. Here we find, however, that traps, even if ideal, disturb the plasma beyond the sheath; this affects incoming ions, and the value of the current [[2]]. The effect clearly vanishes with the grid-E transparency α_E. We

shall take α_E^2 formally small; tipically $\alpha_E^2 \sim 1$, yet the effect of rejected ions proves to be moderately small.

Let the spacecraft wall be the plane $x = 0$ (Fig.1), plasma filling the half-space $x < 0$. The grid $\boldsymbol{E}$ is a circle of radius R centered at $y = z = 0$. For simplicity, we take V_E equal to the spacecraft potential. The potential $V(\bar{r})$ and the ion distribution function $f(\bar{r}, \bar{v})$ obey steady Poisson and Vlasov equations, with the electrons following Boltzmann's law $(e|V_E| >> k_B T_e >> m_e U^2)$,

$$\nabla^2 V = 4\pi e \left[N_\infty \exp\left(\frac{eV}{k_B T_e} \right) - \int f \, d\bar{v} \right],$$

$$\bar{v} \cdot \nabla f - \frac{e}{m_i} \nabla V \cdot \frac{\partial f}{\partial \bar{v}} = 0.$$

We partition f in the form

$$f \equiv f^+ + f^-, \quad f^+(v_x < 0) \equiv 0, \quad f^-(v_x > 0) \equiv 0.$$

The boundary conditions for V, f^+ and f^- are:

$$\text{as } x \longrightarrow -\infty, \quad V \longrightarrow 0, \quad f^+ \longrightarrow f_\infty(\bar{v}) \qquad (1)$$

$$\text{at } x = 0, \quad V = V_E, \quad f^- = H(R - r_\perp)g(\bar{r}_\perp, \bar{v}), \qquad (2)$$

where $f_\infty(\bar{v})$ is a Maxwellian distribution drifting at velocity $\bar{U}$, H is Heavyside's step function (there are no ions leaving the satellite wall) and for the function g we have, for $v_x < 0$, the following expression:

$$g = \alpha_E^2 H \left(\sqrt{2e \frac{V_P - V_E}{m_i}} - |v_x| \right) \times$$

$$\times f^+(x = 0, \bar{r}_\perp, \bar{v}_\perp, |v_x|)$$

(Ions arriving at $\boldsymbol{E}$ with $v_x < \sqrt{2e(V_P - V_E)/m_i}$ emerge from $\boldsymbol{E}$ at the point of entry with equal $\bar{v}_\perp$ and opposite v_x). The Mach number $M \equiv \sqrt{m_i U^2/k_B T_i}$ is moderately large; here we neglect terms of order M^{-2}. Also, we take the ram angle $\theta \leq O(1/M)$, typically leading to $\theta < 20^o$.

Whipple's law corresponds to the limit $\alpha_E^2 \to 0$. Write

$$V = V_0 + \alpha_E^2 V_1 + \dots, \qquad f^\pm = f_0^\pm + \alpha_E^2 f_1^\pm + \dots$$

The lowest order problem is then onedimensional. Since V_E is negative we have $f_0^- \equiv 0$,

$$\frac{d^2 V_0}{dx^2} = 4\pi e \left[N_\infty \exp\left(\frac{eV_0}{k_B T_e} \right) - \int f_0^+ \, d\bar{v} \right],$$

$$v_x \frac{\partial f_0^+}{\partial x} - \frac{e}{m_i} \frac{dV_0}{dx} \frac{\partial f_0^+}{\partial v_x} = 0.$$

The boundary conditions (1) readily determine f_0^+ and V_0. As $x/\lambda_D \to -\infty$, we have $f_0^+ \to f_\infty$ and $V_0/V_E \to 0$. (We have $\int f_\infty(v_x > 0) \, d\bar{v} = 1$, within terms of order $\exp(-M^2/2)$).

The particle flux along x is conserved. The lowest order current I_0 is due to ions with velocity $v_x < \sqrt{2eV_P/m_i}$ outside the sheath,

$$I_0 = \alpha A_E e \int d\bar{v}_\perp \int_{\sqrt{2eV_P/m_i}}^{\infty} v_x f_\infty(\bar{v}) \, dv_x =$$

$$= \alpha A_E e N_0 U \left(\frac{1 - \mathrm{erf}\Delta}{2} + \sqrt{\frac{2}{\pi}} \frac{e^{-\Delta^2}}{2M} \right) \qquad (3)$$

This is Whipple's formula (when using $\cos\theta \approx 1$, to order M^{-2}), where $\alpha = \alpha_E \times \alpha_P \times \alpha_G$ is the overall trap transparency (Fig.1) and

$$\Delta \equiv \left(\frac{eV_P}{k_B T_i} \right)^{1/2} - \frac{M}{\sqrt{2}} \qquad (4)$$

As V_P increases, the parenthesis in (3) goes down from unity to zero, the decrease being centered around $\Delta = 0$.

Terms of order α_E^2 lead now to a correction to Whipple's formula. Inside the sheath, the equation for f_1^- is

$$\left(v_x \frac{\partial}{\partial x} + \bar{v}_\perp \cdot \frac{\partial}{\partial \bar{r}_\perp} \right) f_1^- - \frac{e}{m_i} \frac{dV_0}{dx} \frac{\partial f_1^-}{\partial v_x} = 0$$

To solve it, we ignore the second term (smaller than the first one by a factor λ_D/MR), and use boundary conditions (2) with f_0^+ in g. In the limit $x/\lambda_D \to -\infty$ we find

$$f_1^-(x/\lambda_D \to -\infty) = H(R - r_\perp) \times$$

$$\times H \left(\sqrt{2e \frac{V_P}{m_i}} - |v_x| \right) f_M(-U_x, \bar{U}_\perp) \quad (5)$$

where $f_M(-U_x, \bar{U}_\perp)$ is a Maxwellian distribution drifting at velocity $-U_x, \bar{U}_\perp$.

Outside the sheath, thermal motion spreads disturbances over distances $|x| \sim MR$. The equation for f_1^- is now

$$\left(v_x \frac{\partial}{\partial x} + \bar{v}_\perp \cdot \frac{\partial}{\partial \bar{r}_\perp} \right) f_1^- = 0$$

Taking the Fourier transform with respect to $\bar{r}_\perp$ we readily obtain

$$\tilde{f}_1^-(x, \bar{k}_\perp) \equiv \int f_1^- \exp(-i\bar{k}_\perp \cdot \bar{r}_\perp) \, d\bar{r}_\perp =$$

$$= \tilde{f}_1^-(0) \exp\left(-i \frac{\bar{k}_\perp \cdot \bar{v}_\perp}{v_x} x \right)$$

Here $\tilde{f}_1^-(0)$ is the Fourier transform of $f_1^- (x/\lambda_D \to -\infty)$ as given by (5) above. Integrating $\tilde{f}_1^-$ over $\bar{v}$ and Fourier inverting with respect to $\bar{k}_\perp$ gives N_1^-. At $x = 0$, for instance, we find

$$N_1^- = \frac{1}{2} N_\infty (1 + \mathrm{erf}\Delta) \, H(R - r_\perp)$$

Also, for $-x >> MR$ and $z = 0$, we find

$$N_1^- \approx \frac{N_\infty}{2} \left[\frac{1 + \mathrm{erf}\Delta}{2} + O(\frac{1}{M}) \right] \times$$

$$\times \left(\frac{MR}{x} \right)^2 \exp\left[-\frac{M^2}{2} \left(\theta - \frac{y}{|x|} \right)^2 \right]$$

Outside the sheath f_1^+ obeys (for $v_x > 0$) the equation

$$\left(v_x \frac{\partial}{\partial x} + \bar{v}_\perp \cdot \frac{\partial}{\partial \bar{r}_\perp} \right) f_1^+ =$$

$$= \frac{e}{m_i} \left(\frac{\partial V_1}{\partial x} \frac{\partial}{\partial v_x} + \frac{\partial V_1}{\partial \bar{r}_\perp} \cdot \frac{\partial}{\partial \bar{v}_\perp} \right) f_\infty$$

(here f_0^+ is just f_∞). Fourier transforming with respect to $\bar{r}_\perp$ and using boundary conditions (2) yields $\tilde{f}_1^+$ in terms of $\tilde{V}_1$. Finally, quasineutrality makes the Fourier transformed Poisson equation read

$$0 = N_\infty \frac{e\tilde{V}_1}{k_B T_e} - \int \tilde{f}_1^+ \, d\bar{v} - \tilde{N}_1^- \quad .$$

$\tilde{V}_1(x, \bar{k}_\perp)$ is now given by a linear integral equation defined in the half-space $x < 0$, singular and with a difference kernel (as in the Wiener-Hopf problem). Our equation, however, is of Volterra type. In order to solve it we consider an extended problem: to find $\tilde{V}_1(x, \bar{k}_\perp)$ for $-\infty < x < \infty$, taking for $\tilde{N}_1^-(x > 0)$

the same function $\tilde{N}_1^-(x < 0)$. Fourier transforming with respect to x:

$$\tilde{\tilde{f}}_1^+(k_x, \bar{k}_\perp) = \int_{-\infty}^\infty \tilde{f}_1^+(x, \bar{k}_\perp) \exp(-ik_x x) \, dx$$

and then inverting $\tilde{\tilde{f}}_1^+$ with respect to k_x, we find

$$N_\infty \frac{e\tilde{V}_1(x, \bar{k}_\perp)}{k_B T_i} = \int_{-\infty}^\infty \frac{1}{2\pi} \frac{\exp(ik_x x)}{\beta + L(\zeta)} \tilde{\tilde{N}}_1^-(k_x, \bar{k}_\perp) \, dk_x$$

where $\beta = \dfrac{T_i}{T_e}$, $\zeta \equiv \dfrac{-\bar{k} \cdot \bar{U}}{k\sqrt{2k_B T_i}}$ and

$$\tilde{\tilde{N}}_1^- = \int_{-\infty}^\infty \tilde{N}_1^-(x, \bar{k}_\perp) \exp(-ik_x x) \, dx$$

$$L(\zeta) \equiv \int_0^\infty 2\sigma \exp(-\sigma^2) \exp(2i\zeta\sigma) \, d\sigma \equiv$$

$$\equiv -\frac{1}{2} \frac{dZ(\zeta)}{d\zeta}$$

where $Z(\zeta)$ is the plasma dispersion function.

The solution for $\tilde{V}_1(x, \bar{k}_\perp)$ will be unique for $x < 0$, if unaffected by $\tilde{N}_1^-(x > 0)$, that is, if the condition

$$\int_{-\infty}^\infty \frac{1}{2\pi} \frac{\exp[-ik_x(x' - x)]}{\beta + L(\zeta)} \, dk_x \equiv 0 \qquad x' - x > 0$$

is satisfied. This is indeed the case because the equation $\beta + L(\zeta) = 0$ has no roots in the $\mathrm{Im}\zeta > 0$ half-plane; this is a consequence of quasineutral, ion-acoustic waves being stable, in the Vlasov sense, in the case of a Maxwellian distribution.

To determine the current to order α_E^2, $I = I_0 + \alpha_E^2 I_1$, note that ions entering the sheath within the cylinder $r_\perp < R$, reach the retarding grid $\boldsymbol{P}$ if

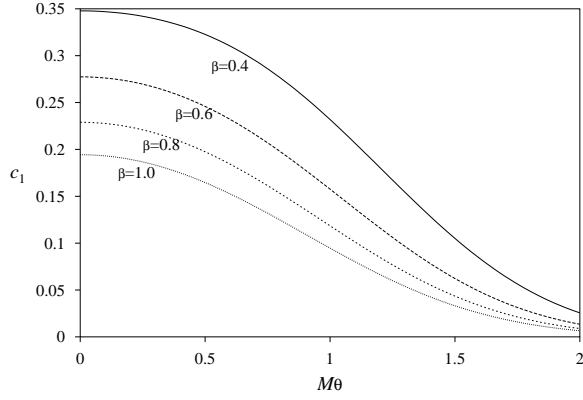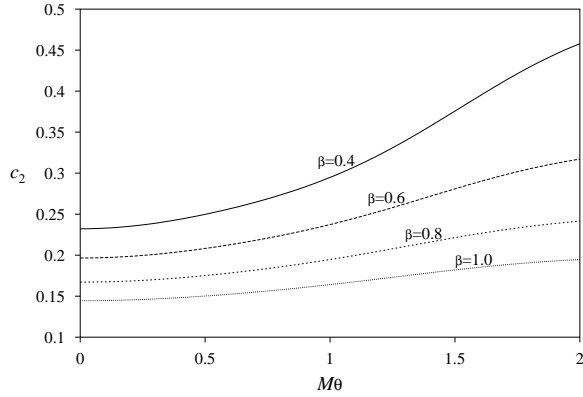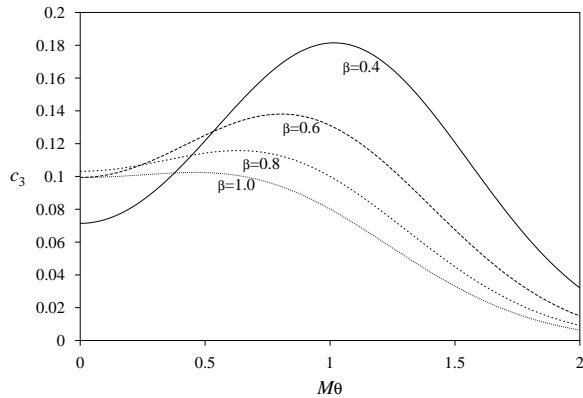$$v_x > \sqrt{\frac{2e[V_P - \alpha_E^2 V_1(x/MR \to 0)]}{m_i}}$$
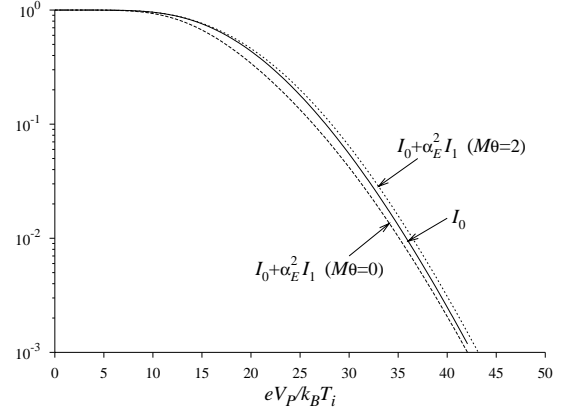
Fig. 2

Fig. 5

Fig. 3

$$I_1 = \alpha e \int_{A_E} d\bar{r}_\perp \int d\bar{v}_\perp \int_{\sqrt{2eV_P/m_i}}^{\infty} dv_x \times$$

$$\times \int \frac{d\bar{k}_\perp}{4\pi^2} \exp(i\bar{k}_\perp \cdot \bar{r}_\perp) \int_{-\infty}^{0} dx' \exp\left(i\frac{\bar{k}_\perp \cdot \bar{v}_\perp}{v_x}x'\right) \times$$

$$\times \frac{e\tilde{V}_1(x')}{m_i} i\bar{k}_\perp \cdot \left(\frac{\partial}{\partial \bar{v}_\perp} - \frac{\bar{v}_\perp}{v_x}\frac{\partial}{\partial v_x}\right) f_\infty$$

Seven of the nine integrals involved can be carried out within terms of order M^{-2}. Using polar coordinates $k_\perp$, ϕ for $\bar{k}_\perp$, we finally arrive at

$$I_1 = -\alpha A_E e N_\infty U \left(c_1 \frac{1 - \mathrm{erf}^2\Delta}{2}\right.$$

$$\left. - \frac{c_2}{M}e^{-\Delta^2}\frac{1 + \mathrm{erf}\Delta}{2} - \frac{c_3}{M}e^{-\Delta^2}\frac{1 - \mathrm{erf}\Delta}{2}\right)$$

where

$$c_j(\beta, M\theta) = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-\infty}^{\infty} \frac{d\zeta e^{-(\zeta + \sqrt{2}M\theta\sin\phi)^2}}{\sqrt{2\pi}(\beta + L(\zeta))} \times h_j$$

with

$$h_1 = \frac{L(\zeta)}{\sqrt{2}}$$

$$h_2 = \frac{[1 - (1 - 2\zeta^2 - \sqrt{2}\zeta M\theta\sin\phi)L(\zeta)]}{\sqrt{\pi}}$$

$$h_3 = \frac{L(\zeta)[1 - 2(\zeta + \sqrt{2}M\theta\sin\phi)(\zeta + (1/\sqrt{2})M\theta\sin\phi)]}{\sqrt{\pi}}$$

Fig. 4

The ion flux through the lateral surface of that cylinder, within the sheath, would lead to corrections of order λ_D/MR. We then obtain

Figures 2-4 give c_j versus $M\theta$ for several values of β. We directly compare I_0 to $I_0 + \alpha_E^2 I_1$, both normalized

to $\alpha A_E e N_\infty U$, in Fig.5, for $\alpha_E^2 = 0.8, M = 6, \beta = 0.5$, and two values of $M\theta$. An approximate comparison between our result and Whipple's results is obtained by using a simple feature of the relation $I(V_P)$, the extremum of dI/dV_P. We find

$$T_i(\text{corrected}) \approx T_i(\text{Whipple}) \times \left(1 - 2\alpha_E^2 \frac{c_2 - c_3}{M}\right).$$

Differences between $T_i(\text{corrected})$ and $T_i(\text{Whipple})$ typically reach $10 - 20\%$ for $\alpha_E^2 \sim 1$.

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