# POISSON SOLVER DOCUMENTATION

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## 1. Introduction

The objectives of this manual is first to provide the background on the finite element methods used in the Poisson solver implemented in SPIS and second to explain the meaning of some numerical parameters that can be tunned by the user.

## 2. Finite element background

In this section, a brief survey of the main principles of the finite element method is given. For more details we refer to the existing litterature (see for example [1]).

2.1. **Electrostatic equations and variationnal formulation.** The electrostatic potential $u$ solves the Poisson equation in a 3D bounded domain $\Omega$ :

$$\begin{cases} -\Delta u = F \; in \; \Omega \\ u = g \; on \; \Gamma_D \\ \dfrac{\partial u}{\partial n} = h \; on \; \Gamma_N \\ \dfrac{\partial u}{\partial n} + \alpha u = r \; on \; \Gamma_R \end{cases} \tag{2.1}$$

where $\dfrac{\partial u}{\partial n}$ is the exterior normal derivative on each point on the boundary of $\Omega$, $\Gamma_D \cup \Gamma_N \cup \Gamma_R$, $r, g, h$ are functions prescribed on the boundary, $F$ the space charge, and $\alpha$ a non negative function.
The boundary conditions have the physical meaning :

- $u = g$ : Dirichlet conditions on $\Gamma_D$ (the conductor part of the spacecraftl where the potential is prescribed)
- $\dfrac{\partial u}{\partial n} = h$ : Neumann condition on $\Gamma_N$ ( the dielectric part of the spacecraftl where the normal component of the electric field is prescribed)
- $\dfrac{\partial u}{\partial n} + \alpha u = r$ : Robin condition on $\Gamma_R$ (the external boundary of the computational domain). Typically $\alpha = 1/R$, where $R$

is the distance between one point of the computational domain and a point of the external boundary. The Robin condition is a transparent condition modelling the behavior of the electrostatic potential far to the plasma.

The classical partial differential equations theory ([4]) provides an equivalence form of equations 2.1 with the following variationnal form :

(2.2)
$$\begin{cases} \displaystyle\int_\Omega \nabla u \nabla v + \int_{\Gamma_R} \alpha u v = \int_\Omega F v + \int_{\Gamma_R} \alpha r v + \int_{\Gamma_N} h v \ \ \forall v,\ v = 0 \ on\ \Gamma_D \\ u = g \ on\ \Gamma_D \end{cases}$$

2.2. **Finite element methods.** The mesh of the computational domain $\Omega$ is a finite set of non overlapping tetrahedrons $T_h$ such that $\Omega = \bigcup_h T_h$ where $T_h \cap T_{h'}$ is empty, reduced to one node of the mesh $(a_i, i = 1, N)$ or to a triangular face. Example of the figure 2.2 illustrates the requirements of the mesh : $T_1 \cap T_2 = \widehat{a_1 a_2 a_4}$ and $T_1 \cap T_3 = a_1$. $V_N$ is the space of piecewise linear continuous functions in each $T_h$. A

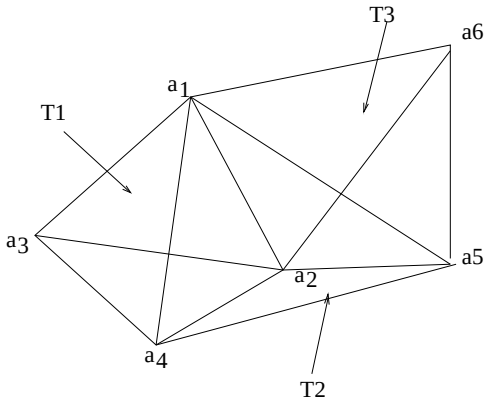


FIGURE 2.1. A picture of three tetrahedrons connected

basis of $V_N$ is obtained by considering the piecewise linear continuous functions $\phi_i$ such that $\phi_i(a_j) = \delta_{ij}$. So, an approximation of $u$ is derived expanding it on this basis : $u(x) \approx \sum_{i=1,N} u_i \phi_i(x)$ Denote by $V_{Nd}$

the subspace of $V_N$ of dimension $N_d$ containing the functions of $V_N$ vanishing on $\Gamma_D$. Using 2.2, a discrete formulation is derived :

(2.3)
$$\sum_{i=1,N} u_i \int_\Omega \nabla\phi_i \nabla\phi_j + \int_{\Gamma_R} \alpha\phi_i\phi_j = \int_\Omega F\phi_j + \int_{\Gamma_R} \alpha i\phi_j + \int_{\Gamma_N} h\phi_j \; \forall \phi_j \in V_{Nd}$$
$$u_i = g(a_i), \forall \phi_i \in V_N - V_{Nd}$$

That can be easily rewritten as a linear algebric system :

(2.4)
$$AU = B$$

where $A$ is a $N_d \times N_d$ matrix, $U$ the vector solution and $B$ the right hand side depending on $f, g, h$ and $r$. $A = (a_{ij})_{1 \le i,j \le N_d}$ with $a_{ij} = \int_\Omega \nabla\phi_i \nabla\phi_j + \int_{\Gamma_R} \alpha\phi_i\phi_j$.

## 3. Linear solver

Here, a brief detail of the linear algebric solver and some explanations about the tunning parameters are provided.

3.1. **Matrix storage.** $\phi_i$ and $\phi_j$ overlap if their corresponding node belong to the same tetrahedron, then $a_{ij} \neq 0$ when $a_j$ and $a_j$ are the nodes of the same tetrahedron $T_h$. This propertie leads to a sparse matrix $A$ that is exploited in the solver. The bandwidth $lb$ of the matrix is defined as the $max_j$ such that $a_{ij} \neq 0,\ j \geq lb\ \forall i$. So, the numbering of the nodes is optimized in order to have the bandwidth minimum using a Cuthill-Makee algorithm, thus the matrix can be efficiently decomposed as Cholesky form with a mimimum storage requirement. In fact, only Cholesky decomposition of a preconditionning of $A$ is performed in order to make efficient the iterative resolution of the linear system. $A$ is stored as a band matrix as it is explained in Lapack documentation [3].

3.2. **Iterative solver.** The linear system is solved by a preconditionned conjuguate gradient (PCG) algorithm [2]. The preconditionning $A_{prec}$ of $A$ is a incomplete LU factorization: it is obtained by eliminating $l_{prec}$ extra-diagonal of $A$. $l_{prec}$ is a parameter $\leq lb$ that can be tunned by the user. It accelerates the convergence of the conjuguate gradient, but is computational cost expensive when $l_{prec}$ is choosen too large. For example $l_{lprec} = 0$, $A_{prec} = diag(A)$ (the default is $l_{prec} = 1$ corresponding to a tridiagonal matrix) ). Each step of the conjuquate gradient requires to solve the linear system $A_{prec}x = y$ that is performed by a call to the java Lapack library containing efficient routines to compute the

solution of a linear system with band matrix. The stop criterium of the algorithm is given by $\epsilon = \frac{||AU - B||}{||B||}$, the default value is $\epsilon = 1d - 6$.

## 4. Poisson nonlinear solver

A specific approach has been also implemented in order to take into account a Boltzmann distribution for electrons. In this model $n_e = n_0^e e^{-\frac{u}{k_B T_e}}$, ($T_e$ is the electronic temperature, $k_B$ the Boltzmann constant and $n_e^0$ the background electron density number). Thus, $F = f - q_e^0 e^{-\frac{u}{k_B T_e}}$ ($f$ is the space charge due to the heaviest charged particles, $q_e^0 = n_e^0/\epsilon_0$). Then the equation 2.1 becomes:

$$\left\{ -\Delta u + q_e^0 e^{-\frac{u}{k_B T_e}} = f \; dans \; \Omega \right. \tag{4.1}$$

Equation 4.1 is rewritten using the variationnal formulation :

$$\left\{ \int_\Omega \nabla u \nabla v + \int_\Omega q_e^0 e^{-\frac{u}{k_B T_e}} v + \int_{\Gamma_R} \alpha u v = \int_\Omega f v + \int_{\Gamma_R} \alpha i v + \int_{\Gamma_N} h v \; \forall v, \; v = 0 \; on \; \Gamma_D \right. \tag{4.2}$$
$$u = g \; on \; \Gamma_D$$

and the discrete formulation :

$$\begin{array}{c} \sum_{i=1,N} u_i \int_\Omega \nabla \phi_i \nabla \phi_j + q_e^0 e^{-\frac{u_j}{k_B T_e}} \int_\Omega \phi_i \phi_j + \int_{\Gamma_R} \alpha \phi_i \phi_j = \\ \int_\Omega f \phi_j + \int_{\Gamma_R} \alpha i \phi_j + \int_{\Gamma_N} h \phi_j \; \forall \phi_j \in V_{Nd} \\ u_i = g(a_i), \forall \phi_i \in V_N - V_{Nd} \end{array} \tag{4.3}$$

that can be rewritten as :

$$AU + M(U) = B \tag{4.4}$$

the nonlinear system is solved by a Newton algorithm with an optimized descent parameter. Each iteration of the Newton algorithm requires a linear system solved by the PCG algorithm described in the previous sections. So, the stop criteria is defined by $\frac{||AU+M(u)-B||}{||B||}$ and the convergence of the algorithm when the residual has reached the stop criterium $\epsilon_1$ (the default value is $1d - 4$).

## 5. Extension to specific geometries

In order to take into account specific geometries like booms or solar panel, new developements have been made. These points are briefly detailed in the following sections.

5.1. **Wire approximation.** Taking conductors with very small radius when compared to the typical dimension of the satellites into account makes the calculation cost high or even impossible. Thus the cost due to the refinement of the mesh in the neighborhood of the boom leads to a considerable increase in the number of tetrahedrons and the simulation time. The idea (see ref [5]) is to construct a boundary condition on the closest nodes to the wire and simulating the presence of the conducting. This boundary condition has the following form:

$$\frac{\partial u}{\partial n}(a_i) + \frac{u(a_i)}{r_i ln(r_i/rad)} = \frac{r}{r_i ln(r_i/rad)}$$

where $r_i$ is the distance from the closest node $i$ to the wire, $rad$ is the radius of the wire and $i$ is the value of the applied potential on the wire (see figure 5.1) This boundary condition is implemented throught the

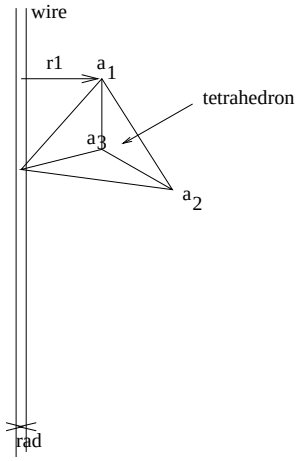


FIGURE 5.1. A wire with one of the closest tetrahedron

same formalism used in the Robin condition.

5.2. **Thin surfaces.** When solar panel are modelled, the thickness of the panel has to be neglected to obtain a reasonnable computational cost. Electrostatic potential is discontinuous across the thin surface generating a singular pertubation at the edge of the surface. A decomposition of the electric potential in a regular part and a singular part is performed in order to extract the discontinuous parts $u = u_{reg} + u_{sing}$. $u_{sing}$ is a global function that appears in the right hand side and behaves like the edge singularity of the laplace operator.
For example (see figure 5.2), if the panel is a plane where the potential

is prescribed to $u_+$ on the superior face and $u_-$ on the inferior face, we have $u_{sing} \approx (u_- - u_+)\dfrac{\theta}{2\pi} + u_+$

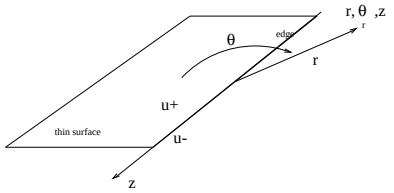


FIGURE 5.2. Solar panel with a polar coordinates system

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